Sometimes it is useful to know you can multiply power series term by term, and without having to worry about radius of convergence issues. This theorem makes it a breeze:

Friday details.

Theorem 5 (Multiplying power series): Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = \left(a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots\right)$$

$$g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n = \left(b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \ldots\right)$$

in $D(z_0; R)$. Then the power series for $f(z)g(z)$ also converges in $D(z_0; R)$ and is given by

$$f(z)g(z) = a_0 b_0 + (a_0 b_1 + a_1 b_0)(z-z_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(z-z_0)^2 + \ldots.$$

$$f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j\, b_{n-j} \right) (z-z_0)^n,$$

in other words, what you expect by formally multiplying and collecting all coefficients for each $(z-z_0)^n$.

proof: We know that power series are Taylor series. Therefore,

do know Taylor series

$$f(z)g(z) = \sum_{n=0}^{\infty} \frac{(fg)^{(n)}(z_0)}{n!} (z-z_0)^n$$

will converge in $D(z_0; R)$. Compute the various derivatives, using the product rule for first, second, ..., n^{th} derivatives of product functions (via induction and the binomial theorem).

$$(fg)(z_0) = a_0 b_0$$

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0) = a_1 b_0 + a_0 b_1$$

$$(fg)''(z_0) = f''(z_0)g(z_0) + 2f'(z_0)g'(z_0) + f(z_0)g''(z_0)$$

$$(fg)''(z_0) = (2\,a_2)b_0 + 2\,a_1 b_1 + a_0(2\,b_2) = 2!(\,a_2 b_0 + a_1 b_1 + a_0 b_2)$$

In general and using the product rule, (checked by induction, as in proof of binomial theorem in first HW),

$$\frac{(fg)^{(n)}(z_0)}{n!} \qquad (fg)^{(n)}(z_0) = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(z_0)\, g^{(n-j)}(z_0)$$

$$= \sum_{j=0}^{n} \frac{n!}{\cancel{j}!\,(n\cancel{-}j)!} (\cancel{j}!\, a_j)\ (n\cancel{-}j)!\, b_{n-j} = \frac{\cancel{n!}}{\cancel{n!}} \sum_{j=0}^{n} a_j\, b_{n-j}$$

$\binom{n}{j}$ $\qquad a_j = \frac{f^{(j)}(z_0)}{j!}$ $\qquad b_{n-j} = \frac{g^{(n-j)}(z_0)}{(n-j)!}$

Q.E.D.

Name: Solution

Math 4200 Quiz week 10 October 28, 2020

1a) Find the first three non-zero terms of the Taylor expansion of

$c_1 z + c_3 z^3 + c_5 z^5 + \cdots$

$$\tan(z) = \frac{\sin(z)}{\cos(z)} \iff (\cos z)(\tan z) = (\sin z)$$

at $z_0 = 0$. Hint: Use the trick about multiplying Taylor series that we discuss at the end of class. Since $\tan(z)$ the product of an odd function times an even function, it's an odd function, so only the coefficients of the odd powers of z can be non-zero.

(8 points)

using power series for $\cos z$ & $\sin z$ we have

$$\left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \ldots\right)\left(c_1 z + c_3 z^3 + c_5 z^5 + \ldots\right) = z - \frac{z^3}{6} + \frac{z^5}{120} \cdots$$

$4! = 24$
$5! = 120$
$6! = 720$

match coefficients:

z^1: $c_1 = 1$

z^3: $-\frac{1}{2}c_1 + c_3 = -\frac{1}{6} \Rightarrow c_3 = -\frac{1}{6} + \frac{1}{2} = \frac{1}{3}$

z^5: $c_5 - \frac{1}{2}c_3 + \frac{1}{24}c_1 = \frac{1}{120} \Rightarrow c_5 = \frac{1}{120} + \frac{1}{2}\cdot\frac{1}{3} - \frac{1}{24} = \frac{1+20-5}{120} = \frac{16}{120} = \frac{2}{15}$

$$\boxed{\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \ldots}$$

easy to make algebra error!

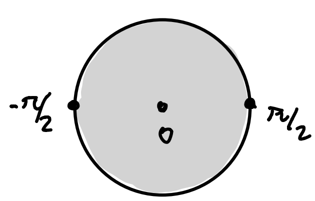
technology! $\tan z = z + \frac{1}{3}z + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \frac{62}{2835}z^9 + \ldots$!

1b) What is the radius of convergence of the series in 1a)? You can answer this without actually knowing the Taylor series!!

(2 points)

$\tan z = \frac{\sin z}{\cos z}$ $\cos z$ has zeroes at $\frac{\pi}{2} + k\pi$ $k \in \mathbb{Z}$.
(where $\sin z = \pm 1$).

So use Theorem 4!



$\tan z$ is analytic in $D(0; \pi/2)$
and cannot be defined to be analytic at $\pm\pi/2$ since $\lim_{z \to \pm\pi/2} |\tan z| = +\infty$.

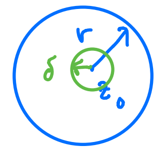
$\Rightarrow R = \pi/2$

Math 4200
Friday October 30
3.2-3.3 isolated zeroes theorem, uniqueness of analytic extensions; begin 3.3 Laurent series. We'll start today by quickly explaining the theorem about multiplying Taylor series in Wednesday's notes, that you used succesfully on the quiz.

Announcements:

Hw?
2nd midterm in 2 weeks!
projects?

Consequences of power series for analytic functions:



$f(z_0) = 0.$

Theorem (Isolated zeroes theorem). Let

- $A \subseteq \mathbb{C}$ be an open, connected set,
- $f: A \to \mathbb{C}$ analytic,
- $D(z_0; r) \subseteq A$,
- $f(z_0) = 0$.

Then either $f(z) \equiv 0$ in $D(z_0; r)$ or $\exists\, \delta > 0$ such that $f(z) \neq 0$ $\forall\; 0 < |z - z_0| < \delta$.

proof: f has convergent Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad\qquad |z - z_0| < r.$$

If all the $a_n = 0$ then $f \equiv 0$ in $D(z_0; r)$. Otherwise let a_N be the first non-zero coefficient in the power series, so

$a_N \neq 0$

$$f(z) = \sum_{n=N}^{\infty} a_n (z - z_0)^n \quad = a_N (z-z_0)^N + a_{N+1}(z-z_0)^N + \ldots$$

and factor out the lowest power of $(z - z_0)$ that appears in this series:

$$f(z) = (z - z_0)^N \left(a_N + \sum_{n=N+1}^{\infty} a_n (z - z_0)^{n-N} \right)$$

$$\boxed{f(z) = (z - z_0)^N g(z)} \qquad g(z_0) = a_N.$$

where $g(z_0) = a_N \neq 0$ and $g(z)$ is analytic and hence continous near z_0. Thus there exists $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow g(z) \neq 0$. This proves the claim.

QED.

$|g(z_0)| = |a_N| \neq 0$

$\exists\, \delta$ s.t. $|z - z_0| < \delta \Rightarrow |g(z) - g(z_0)| < \frac{1}{2}|a_N|$.

$\Rightarrow |g(z)| = |g(z_0) - (g(z_0) - g(z))|$ RT I

$> |a_N| - \frac{1}{2}|a_N| = \frac{1}{2}|a_N|$ ∎

There is a surprising consequence of the isolated zeroes theorem:

<u>Corollary</u> (Unique extensions theorem) Let $A \subseteq \mathbb{C}$ be open and connected; $f, g : A \to \mathbb{C}$ analytic. Supposed there exists

$$\cancel{A \subseteq \mathbb{C} \text{ be open and conne}}\text{cted},$$

$$\cancel{f, g : A \to \mathbb{C} \text{ analy}}\text{tic};$$

!!!

$$\{z_k\} \subseteq A,\ \{z_k\} \to z_0 \in A,\ z_k \neq z_0,\ k \in \mathbb{N}.$$

$z_k \quad z_0$

$$f(z_k) = g(z_k)\ \forall\ k$$ (also $f(z_0) = g(z_0)$ by continuity).

Then $f(z) = g(z)\ \forall\ z \in A$.

proof: $f - g : A \to \mathbb{C}$ is analytic and $(f-g)(z_k) = 0\ \forall\ k$. Thus z_0 is a zero of $f - g$ which is not isolated. Thus by the isolated zeroes theorem,

$$(f-g)(z) \equiv 0,\ \forall\ z \in \mathrm{D}(z_0; r) \subseteq A.$$

(This is already surprising.) Now, consider

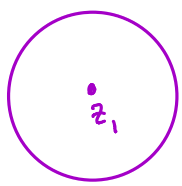
$$B := \{z \in A \mid (f-g)^{(n)}(z) = 0\ \forall\ n = 0, 1, 2, \ldots\}.$$

We have $\mathrm{D}(z_0; r) \subseteq B$ since $(f-g)(z) \equiv 0$ in $\mathrm{D}(z_0; r)$.

- B is closed in A because if $\{w_k\} \subseteq B$, $\{w_k\} \to w \in A$ then $0 = (f-g)^{(n)}(w_k) \to (f-g)^{(n)}(w)\ \forall\ n$.

- B is open in A because if $z_1 \in B$ the Taylor series for f at z_1 is the zero function, so for any $\mathrm{D}(z_1; r) \subseteq A$ we also have $\mathrm{D}(z_1; r) \subseteq B$.

Thus, since A is connected, $B = A$.



■

Example (also see one of your homework problems). It is not clear without a lot of work why the Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \mathrm{Re}\, s > 1.$$

can be extended as an analytic function (with different formulas), beyond the plane $\mathrm{Re}(s) > 1$ on which the series converges. But in fact, it can be extended as an analytic function on $\mathbb{C} \setminus \{1\}$. The Unique extensions theorem says there's only one possible extension.

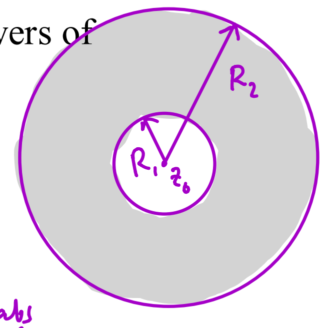
3.3 Laurent series. If $f(z)$ is an analytic function on a punctured disk or on an annulus centered at z_0, then f can be expressed as a power series expansion using non-negative *and* negative powers of $(z - z_0)$. These series are called *Laurent series*

Laurent Series Theorem For $0 \le R_1 < R_2$ let

$$A = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$$

be an open annulus (or punctured disk in case $R_1 = 0$). Then (1) and (2) below are equivalent, and the uniqueness of Laurent coefficients (3) also holds:

- (1) $f : A \to \mathbb{C}$ is analytic.
- (2) $f(z)$ has a power series expansion using non-negative and negative powers of $(z - z_0)$:



$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{a_{-m}}{(z - z_0)^m}.$$

$$:= S_1(z) + S_2(z).$$

- Here $S_1(z)$ converges for $|z - z_0| < R_2$ and uniformly absolutely for $|z - z_0| \le r_2 < R_2$. And $S_2(z)$ converges for $|z - z_0| > R_1$, and uniformly abs for $|z - z_0| \ge r_1 > R_1$.

(3) The Laurent coefficients a_k, $k \in \mathbb{Z}$ are uniquely determined by f. Specifically, if γ is any p.w. C^1 contour in A, with $I(\gamma, z_0) = 1$, e.g. any circle of radius r, with $R_1 < r < R_2$, then

$$a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta$$

In particular the contour integral of f itself has value

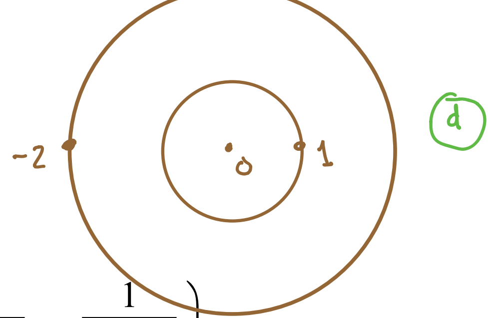
$$\int_\gamma f(\zeta) d\zeta = 2\pi i \, a_{-1}.$$

For this reason, the coefficient a_{-1} of $\frac{1}{z - z_0}$ in the Laurent series, is called the *residue* of f at z_0. (Because it's the only part of the Laurent series you need to know in order to compute the contour integral of f.)

Note: (2) $\Rightarrow$ (1) of the theorem is immediate, since uniform limits of analytic functions are analytic. We'll prove (2) $\Rightarrow$ (3) in today's notes, and then do (1) $\Rightarrow$ (2) on Monday. It will rely on geometric series magic, as did our theorem about Taylor series for analytic functions in disks.

Examples:

$\boxed{\frac{1}{1-\omega}}$



1) Consider

$$f(z) = \frac{1}{(z-1)(z+2)} = \frac{1}{3}\left(\frac{1}{z-1} - \frac{1}{z+2}\right).$$

Find the following series expansions for $\underline{z_0 = 0}$:

a) Taylor series for $|z| < 1$ •

b) Laurent series for $1 < |z| < 2$. •

c) Laurent series for $|z| > 2$. •

$$f(z) = \frac{1}{3}\left(-\frac{1}{1-z}\right) - \frac{1}{3}\frac{1}{2}\frac{1}{1-(-\frac{z}{2})}$$

$$= -\frac{1}{3}\sum_{n=0}^{\infty} z^n - \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n} z^n$$

$R=1$ $\quad$ $R=2$

d) Use residues from the Laurent series to compute $\int_\gamma f(z)\,dz$ for the three index-one circles centered at the origin, of radii $\frac{1}{2}, \frac{3}{2}, 3$. Notice that this is reproducing results you already know how to find using the Cauchy integral formula and other means.

b) $1 < |z| < 2$ $\quad$ $f(z) = \frac{1}{3}\frac{1}{z-1} - \frac{1}{3}\frac{1}{z+2} = \frac{1}{3}\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) - \frac{1}{3}\frac{1}{z+2}$

big || term is $|z|$.

$|z|>1$
$|\frac{1}{z}|<1$

$$= \frac{1}{3}\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n} z^n$$

conv for $|z|>1$ $\quad$ conv for $|z|<2$

c) $|z|>2$. $\quad$ $f(z) = \frac{1}{3}\frac{1}{z-1} - \frac{1}{3}\frac{1}{z+2}$

use b) $\quad$ big term is z!

$$-\frac{1}{3}\frac{1}{z}\left(\frac{1}{1+2/z}\right)$$

$|z|>2$
$|\frac{2}{z}|<1$.

$$\frac{1}{1+2/z} = \frac{1}{1-(-2/z)}$$

$$\Rightarrow \; * \; f(z) = \frac{1}{3}\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{3}\frac{1}{z}\left(\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{z^n}\right)$$

$f(z) = \frac{1}{3}\frac{1}{z-1} - \frac{1}{3}\frac{1}{z+2}$ $\quad$ $|z|>1$ $\quad$ $|z|>2$.

d) $\oint_{|z|=1/2} f(z)\,dz = 0$ Cauchy's Thm

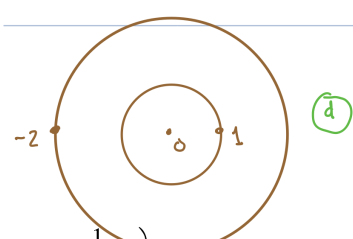
$\oint_{|z|=3/2} f(z)\,dz = \frac{1}{3}2\pi i$ $\quad$ Deformation thm

Use Laurent series b) interchange $\int\sum \sim \sum\int$

$\int_\gamma z^n\,dz = 0$ by FTC except for $n=-1$

1st sum $\frac{1}{z}$ term is $\frac{1}{3}\frac{1}{z}$

its int. is $\frac{1}{3}\cdot 2\pi i$ ✓



$\oint_{|z|=3} f(z)\,dz = \frac{1}{3}\cdot 2\pi i - \frac{1}{3}\cdot 2\pi i = 0$.

$- \frac{1}{z+2}\Big).$

continue on Monday!

2a) What is the Laurent series for $z\,e^{\frac{1}{z}}$ in $\mathbb{C} \setminus \{0\}$?
2b) What is the value of

$$\int_\gamma z\,e^{\frac{1}{z}}\,dz$$

if γ is a closed contour in $\mathbb{C} \setminus \{0\}$, with $I(\gamma;\,0) = 1$?

proof of (2) ⇒ (3) in the Laurent series theorem:

(2) $f(z)$ has a power series expansion using non-negative and negative powers of $(z - z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad + \sum_{m=1}^{\infty} a_{-m} (z - z_0)^{-m}$$
$$:= S_1(z) + S_2(z).$$

Here $S_1(z)$ converges for $|z - z_0| < R_2$ and uniformly absolutely for $|z - z_0| \le r_2 < R_2$. And $S_2(z)$ converges for $|z - z_0| > R_1$, and uniformly for $|z - z_0| \ge r_1 > R_1$.

(3) The Laurent coefficients a_k, $k \in \mathbb{Z}$ are uniquely determined by f. Specifically, if γ is any p.w. C^1 contour in A, with $I(\gamma, z_0) = 1$, e.g. any circle of radius r, with $R_1 < r < R_2$, then

$$a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta$$

In particular the contour integral of f itself has value

$$\int_\gamma f(\zeta) d\zeta = 2\pi i\, a_{-1}.$$

proof: We'll write $f(\zeta) = S_1(\zeta) + S_2(\zeta)$ and just compute the contour integrals above. We'll use the uniform convergence of the series $S_1(\zeta)$, $S_2(\zeta)$ on γ to interchange the integrals with the summations:

Math 4200-001
Week 10-11 concepts and homework
3.2-3.3
Due Friday November 6 at start of class.

3.3 1ab, 4, 6, 8, 13, 15, 17, 18, 19, 20

w10.1a) Use the definitions of even and odd functions to show that if f is analytic in a neighborhood of the origin and if f is even, then its Taylor series at $z_0 = 0$ only contains even powers of z; and if f is odd the Taylor series only contains odd powers.
w10.1b) Are the same facts true for the Laurent series based at $z_0 = 0$, for even and odd analytic functions defined in annuli concentric to the origin?

uniqueness of extensions.
w10.2) Let f be an entire function. Suppose $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ for all positive integers n.
Is it possible for $f(-1)$ to equal -1? Explain.
w10.3) Use power series or L'Hopital's rule to find
$$\lim_{z \to 0} \frac{\cos(z) - 1}{z \sin(z)}$$
w10.4) Continuing the text problem 3.3.4, find the Laurent series for
$$\frac{1}{z(z-1)(z-2)}$$
valid for $|z| > 2$.

w10.4) Which of these functions has a removable singularity at $z = 0$?

a) $\dfrac{\cos(z) - 1}{z \sin(z)}$ (see w10.2)

b) $\dfrac{\cos(z) - 1}{z^3}$.